Poiseuille Flow of a Power–Law Fluid between Coaxial Cylinders

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SYNOPSIS

An analytical expression is derived, which permits direct evaluation of the volumetric flow rate or the pressure gradient for the laminar flow of a power–law fluid in a concentric annulus. The derivation is valid for tangential as well as axial flow, provided a parameter q, in the problem formulation, is substituted by appropriate binary values. The final flow rate solution is in the form of an elegant algebraic equation (not a definite integral) and holds for all values of the power–law index (not merely reciprocal integers). Attention is focused on the tangential pressure flow problem, where the zero-shear plane does not coincide with the location of the maximum velocity. © 1992 John Wiley & Sons, Inc.

INTRODUCTION

As the solutions to flow problems of non-Newtonian fluids between coaxial cylinders are often complex, Worth[1] investigated the accuracy of approximating such flows by an equivalent, parallel-plate geometry in four situations: tangential drag flow, axial drag flow, tangential pressure flow, and axial pressure flow. The parallel-plate analogy could result in considerable errors, for it is rigorously valid from a mathematical viewpoint only in the limit of the radius ratio of the coaxial cylinders tending to unity. Hence, it would be beneficial if the exact solutions in cylindrical coordinates were available for the four flow cases of importance. Worth[1] has presented exact, analytical solutions[2] for the two drag flow situations. However, he has expressed the volume rate of flow for the two pressure flow situations in the form of definite integrals, whose evaluation requires numerical quadrature in general. Though exact power-series expansion solutions are possible for the case where the power–law index is a reciprocal integer, such solutions are of limited practical utility and are often cumbersome. The only analytical solution provided by Worth[1] for these pressure flow situations is for the Newtonian case.

The aim of this article is to establish an exact analytical expression for the volumetric flow rate during the flow of a power–law fluid under the influence of a pressure gradient in the tangential or the axial direction. Such a solution would be useful in the analyses of some polymer processes because, as pointed out by Worth,[1] tangential pressure flow occurs in spiral molds and axial pressure flow in pipe-extrusion dies. The formulation below is presented as an attempt to eliminate the necessity of solving the two flow situations separately.

THEORETICAL FORMULATION

Under consideration is a steady, laminar, incompressible pressure flow in an isothermal system, consisting of two long, coaxial cylinders with negligible end effects. The theoretical formulation is generalized, in that it applies to both tangential flow [Fig. 1(a)] and axial flow [Fig. 1(b)], provided that the quantities are defined appropriately as shown in Table I.

The equation of motion on neglecting inertial effects can be reduced as shown:

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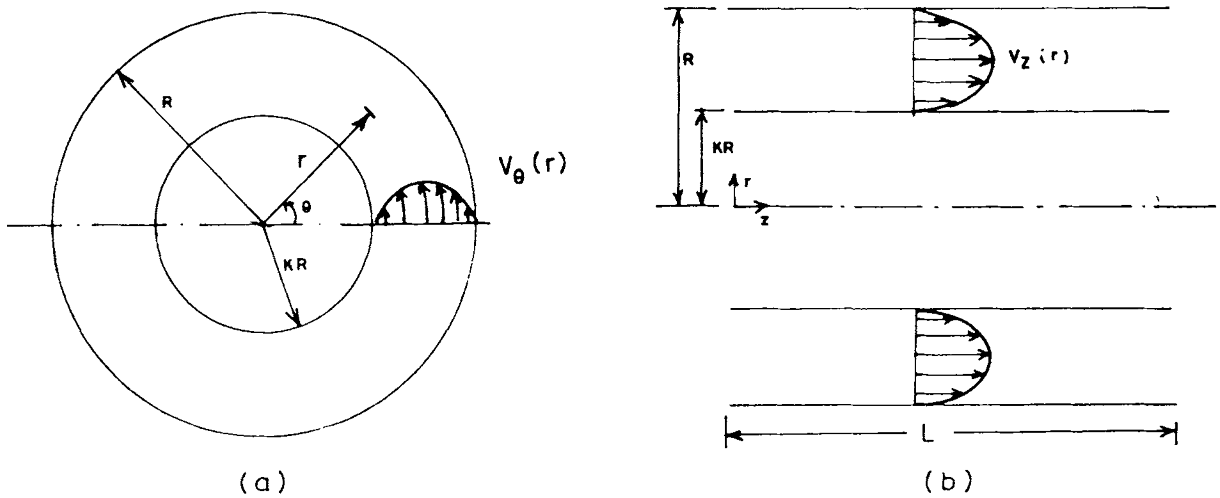

Figure 1 Schematic diagram of pressure flow between coaxial cylinders: (a) tangential flow; (b) axial flow.

$$\frac{1}{r^2}\frac{d}{dr}(r^2\tau_{r\theta}) = -\frac{1}{r}\frac{dP}{d\theta} \quad \text{for tangential flow and} \quad (1a)$$

$$\frac{1}{r}\frac{d}{dr}(r\tau_{rz}) = -\frac{dP}{dz} \quad \text{for axial flow.} \quad (1b)$$

Equation (1) may be integrated to yield the shear stress in the following, dimensionless form (using $q = 0$ and $q = 1$ for tangential and axial flows, respectively):

$$\tau^* = \xi^q - \lambda^2/\xi^{2-q} \quad (2)$$

where $\xi = r/R$ is the dimensionless radial distance. Here, λ is the dimensionless constant of integration and corresponds to the position of the zero-shear plane (i.e., $\tau^* = 0$ at $\xi = \lambda$).

The rheological behavior of the inelastic, non-Newtonian fluid is described by the Ostwald–de Waele power–law model:

$$\tau = -m|\gamma|^{n-1}\gamma \quad (3)$$

where the shear rate γ is given, in general for the two flow situations, by $r^{1-q}d(v/r^{1-q})/dr$. The above rheological equation must be separately adapted[3,4,5] for the two regions inside and outside of the zero-shear plane (denoted by subscripts i and o, respectively) after accounting for the sign of γ, a fact ignored by Worth.[1] Thus, on combining eq. (2) with the dimensionless form of eq. (3) (namely, $\tau^* = -|\gamma^*|^{n-1}\gamma^*$), we obtain

$$\gamma_i^* = (\lambda^2/\xi^{2-q} - \xi^q)^{1/n} \quad \text{for } K \le \xi \le \lambda \quad (4a)$$

Table I Notation for Tangential and Axial Annular Pressure Flows

Quantity	Tangential Flow	Axial Flow
Parameter q	0	1
Shear stress τ	$\tau_{r\theta}$	τ_{rz}
Dimensionless shear stress τ^*	$\tau_{r\theta}[(-dP/d\theta)/2]^{-1}$	$\tau_{rz}[(-dP/dz)R/2]^{-1}$
Shear rate γ	$\gamma_{r\theta} = rd(v_\theta/r)/dr$	$\gamma_{rz} = dv_z/dr$
Characteristic shear rate γ_c	$[(-dP/d\theta)/(2m)]^{1/n}$	$[(-dP/dz)R/(2m)]^{1/n}$
Dimensionless shear rate γ^*	$\gamma_{r\theta}/\gamma_c$	γ_{rz}/γ_c
Velocity v	v_θ	v_z
Dimensionless velocity v^*	$v_\theta/(R\gamma_c)$	$v_z/(R\gamma_c)$
Dimensionless volumetric flow rate Q^*	$2Q/(LR^2\gamma_c)$	$Q/(\pi R^3\gamma_c)$

and

$$\gamma_o^* = -(\xi^q - \lambda^2/\xi^{2-q})^{1/n} \quad \text{for } \lambda \le \xi \le 1. \quad (4b)$$

As $\gamma^* = \xi^{1-q} d(v^*/\xi^{1-q})/d\xi$, the dimensionless local velocities in the two regions obtained on integration are

$$v_i^* = \xi^{1-q} \int_K^{\xi} [(\lambda^2/x^{2-q} - x^q)^{1/n} x^{q-1}]\, dx \quad \text{for } K \le \xi \le \lambda \quad (5a)$$

and

$$v_o^* = \xi^{1-q} \int_{\xi}^{1} [(x^q - \lambda^2/x^{2-q})^{1/n} x^{q-1}]\, dx \quad \text{for } \lambda \le \xi \le 1. \quad (5b)$$

The no-slip boundary condition has been utilized at both stationary surfaces in obtaining eq. (5).

The equation for determining λ is obtained by imposing the condition of the velocity being continuous at $\xi = \lambda$ on eq. (5). Thus,

$$\int_K^{\lambda} (\lambda^2/\xi^{2-q} - \xi^q)^{1/n} \xi^{q-1}\, d\xi - \int_{\lambda}^{1} (\xi^q - \lambda^2/\xi^{2-q})^{1/n} \xi^{q-1}\, d\xi = 0 \quad (6)$$

The quantity of practical interest is the volumetric flow rate, which may be expressed in dimensionless form as:

$$Q^* = 2 \int_K^1 \xi^q v^*\, d\xi \quad (7)$$

The above integral may be evaluated analytically, as demonstrated in the next section.

EXACT ANALYTICAL EXPRESSION FOR VOLUMETRIC FLOW RATE

Equation (7) may be integrated by parts and the boundary conditions utilized to obtain

$$Q^* = -2/(q+1) \int_K^1 \xi^{q+1} (dv^*/d\xi)\, d\xi \quad (8)$$

The above equation may be rewritten using eq. (7) as:

$$Q^* = -\int_K^1 \xi^2 [d(v^*/\xi^{1-q})/d\xi]\, d\xi \quad (9)$$

Substituting eq. (4) in eq. (9), we obtain

$$Q^* = -\int_K^{\lambda} (\lambda^2/\xi^{2-q} - \xi^q)^{1/n} \xi^{q+1}\, d\xi + \int_{\lambda}^{1} (\xi^q - \lambda^2/\xi^{2-q})^{1/n} \xi^{q+1}\, d\xi \quad (10)$$

Integrating eq. (10) by parts yields

$$Q^* = n/(2n+2) \Big[(1-\lambda^2)^{1/n+1} - K^{q+(q-2)/n} (\lambda^2 - K^2)^{1/n+1} - (q + (q-2)/n) \times \int_K^1 |\lambda^2 - \xi^2|^{1/n+1} \xi^{q-1+(q-2)/n}\, d\xi \Big] \quad (11)$$

Using eq. (6), eq. (10) can also be written as:

$$Q^* = \int_K^1 |\lambda^2 - \xi^2|^{1/n+1} \xi^{q-1+(q-2)/n}\, d\xi \quad (12)$$

Combining eqs. (11) and (12), an elegant analytical expression for the volumetric flow rate, valid for all values of n and K, is obtained as given below:

$$Q^* = (2 + q + q/n)^{-1} [(1-\lambda^2)^{1/n+1} - K^{q+(q-2)/n} (\lambda^2 - K^2)^{1/n+1}] \quad (13)$$

The above derivation draws on some concepts first proposed by Hanks and Larsen,[3] and later used by Malik and Shenoy.[4] The derivation is, however, different, in that the route adopted by Hanks and Larsen[3] is more complicated and involves two iterated integrals. Further, Malik and Shenoy,[4] who studied generalized annular couette flow, had a simpler shear rate expression; they were concerned only with axial flow and hence did not require an intermediate step in their derivation, such as the one in eq. (9) above.

RESULTS

Tangential Annular Pressure Flow of a Power-Law Fluid

No analytical solution appears to exist in the literature for this case. It may be simply obtained by setting $q = 0$ in eq. (13). Thus,

$$Q = (LR^2/4)[(-dP/d\theta)/(2m)]^{1/n} \times [(1-\lambda^2)^{1/n+1} - K^{-2/n}(\lambda^2 - K^2)^{1/n+1}] \quad (14)$$

The values of λ, to be used in the above equation, are given in Table II for various K and n. They were computed by iteratively solving eq. (6) with $q = 0$ using the Newton–Raphson technique. The quadrature routine QDAGS, available in IMSL,[6] was used to evaluate numerically the integrals involved.

Knowing K and n, the value of λ may be interpolated from Table II and may be subsequently used in eq. (14) to calculate Q if $(-dP/d\theta)$ is specified or $(-dP/d\theta)$ if Q is specified. Figure 2 shows the plot of the dimensionless, volumetric flow rate Q^* against the radius ratio K for various values of n from 0.1 to 1.0.

Tangential Annular Pressure Flow of a Newtonian Fluid

For $n = 1$ and $q = 0$, eqs. (6) and (14) yield:

$$\lambda^2 = 2K^2 \ln K/(K^2 - 1) \quad (15a)$$

and

$$Q = [LR^2/(2\mu)](-dP/d\theta) \times [(1-K^2)/4 - (K \ln K)^2/(1-K^2)] \quad (15b)$$

Equation (15a) may be used to compute the λ-values in the last row in Table II.

Tangential Annular Pressure Flow of a Power–Law Fluid with *n* Being Reciprocal Integers

Analytical expressions for v^* and λ may be formally obtained from eqs. (5) and (6), when the reciprocal of n is an integer, by binomial expansion, following the methodology of Fredrickson and Bird.[7] Typically, these expressions are cumbersome and may not permit easy calculation of λ. Even for the simplest case (aside from the Newtonian), where $n = 0.5$, the λ-equation is

$$\lambda^4(1 + K^4) - 4\lambda^2K^2(1 + K^2) + 4K^4 \ln(\lambda^2/K) + 6K^4 = 0 \quad (16)$$

In Table II, the λ values for $n = 0.5$ conform with the above equation.

Axial Annular Pressure Flow of a Power–Law Fluid

Fredrickson and Bird[7] first studied this problem, and their result is given in eq. (12) by setting $q = 1$. As they could not reduce the integral in eq. (12) for arbitrary n, they evaluated the integral by power

Table II Values of $\lambda(K, n)$ for Tangential Annular Pressure Flow

	λ								
n	$K = 0.1$	$K = 0.2$	$K = 0.3$	$K = 0.4$	$K = 0.5$	$K = 0.6$	$K = 0.7$	$K = 0.8$	$K = 0.9$
0.10	0.1531	0.2943	0.4235	0.5399	0.6435	0.7349	0.8151	0.8853	0.9465
0.15	0.1583	0.3013	0.4304	0.5457	0.6478	0.7378	0.8167	0.8860	0.9467
0.20	0.1632	0.3077	0.4366	0.5508	0.6516	0.7403	0.8181	0.8866	0.9468
0.25	0.1679	0.3136	0.4422	0.5554	0.6550	0.7424	0.8193	0.8871	0.9469
0.30	0.1722	0.3190	0.4472	0.5595	0.6579	0.7443	0.8204	0.8875	0.9470
0.35	0.1764	0.3240	0.4518	0.5632	0.6606	0.7460	0.8213	0.8879	0.9471
0.40	0.1803	0.3286	0.4560	0.5665	0.6629	0.7475	0.8221	0.8883	0.9472
0.45	0.1841	0.3330	0.4599	0.5695	0.6651	0.7489	0.8229	0.8886	0.9473
0.50	0.1877	0.3370	0.4635	0.5723	0.6670	0.7501	0.8236	0.8889	0.9474
0.55	0.1911	0.3403	0.4668	0.5749	0.6688	0.7512	0.8242	0.8892	0.9474
0.60	0.1943	0.3443	0.4698	0.5772	0.6704	0.7523	0.8247	0.8894	0.9475
0.65	0.1974	0.3477	0.4726	0.5794	0.6719	0.7532	0.8252	0.8896	0.9475
0.70	0.2004	0.3508	0.4753	0.5814	0.6733	0.7541	0.8257	0.8898	0.9476
0.75	0.2032	0.3537	0.4778	0.5832	0.6746	0.7549	0.8261	0.8900	0.9476
0.80	0.2059	0.3565	0.4801	0.5849	0.6758	0.7556	0.8265	0.8902	0.9477
0.85	0.2085	0.3592	0.4822	0.5866	0.6769	0.7563	0.8269	0.8903	0.9477
0.90	0.2110	0.3616	0.4843	0.5881	0.6779	0.7569	0.8273	0.8905	0.9477
0.95	0.2134	0.3640	0.4862	0.5895	0.6789	0.7575	0.8276	0.8906	0.9478
1.0	0.2157	0.3662	0.4880	0.5908	0.6798	0.7581	0.8279	0.8907	0.9478

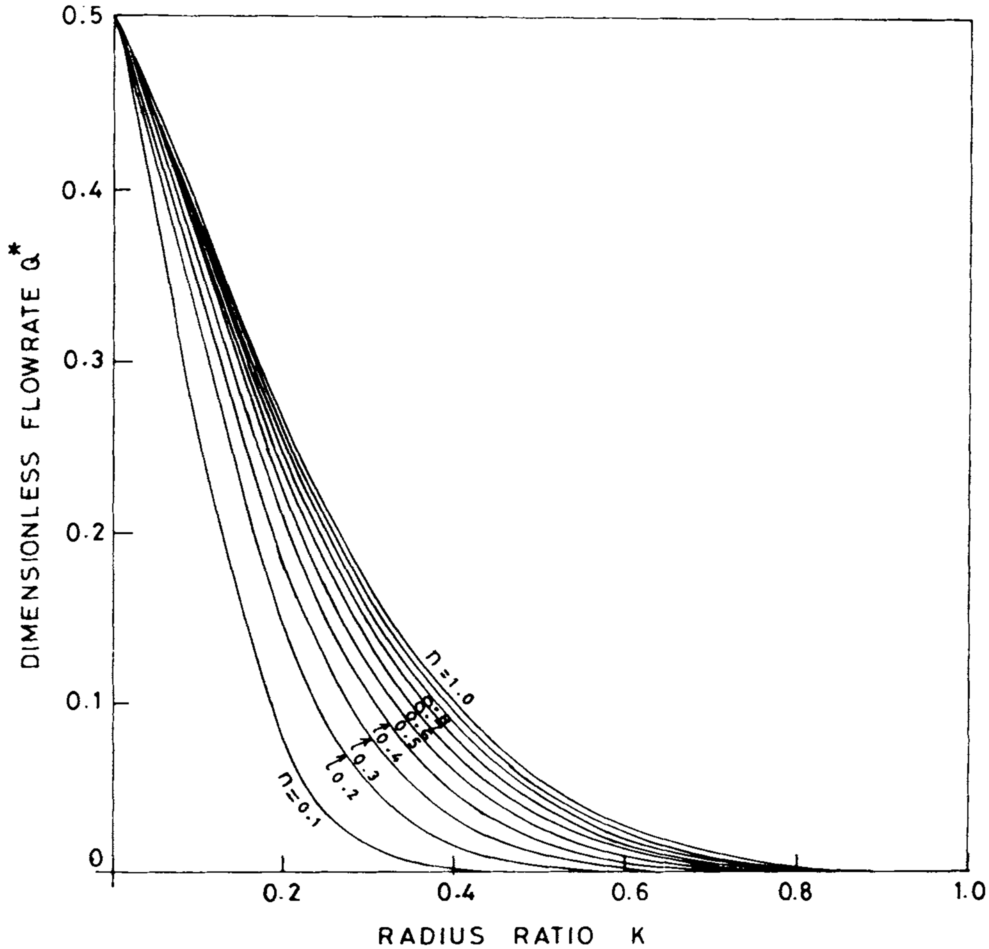

Figure 2 Dimensionless volumetric flow rate vs. radius ratio for tangential pressure flow.

series expansion for cases where $(1/n)$ was an integer.

Clearly, the useful equation for the volume rate of flow in this case is eq. (13) with $q = 1$, which is the result of Hanks and Larsen.[3] The required values of λ have been tabulated by them for various K and n, by solving eq. (6) with $q = 1$.

DISCUSSION

For tangential pressure flow, Worth[1] has stated that "the constant of integration (λ) is related to the coordinate of the plane of zero shear stress (maximum velocity)." Though the plane of maximum velocity coincides with the plane of zero shear stress in the case of axial flows, it is important to note that this is not the case for tangential flows. This may be argued simply as follows.

The shear stress is zero at that plane where the shear rate is zero. Using the relevant expression for γ in Table I for the tangential flow geometry, it is observed that the shear stress is zero at that value of ξ where $dv_\theta/d\xi = v_\theta/\xi$. On the other hand, the maximum in the velocity profile occurs at that value of ξ where $dv_\theta/d\xi = 0$. Since v_θ is non-zero everywhere within the region $(K < \xi < 1)$, the two conditions are never satisfied simultaneously.

This is clearly observed in Figure 3, which shows the variation in the velocity profile with n (generated by numerical integration of eqs. (5) for a radius ratio of 0.5 and $q = 0$). The zero-shear plane never corresponds to the location of the maximum velocity for tangential pressure flow of a pseudoplastic fluid, though they move toward each other with increasing n. As $dv_\theta/d\xi$ $(=v_\theta/\xi)$ must be positive at the plane of zero-shear, it necessarily lies between the inner cylinder and the plane of maximum velocity (see Fig. 3).

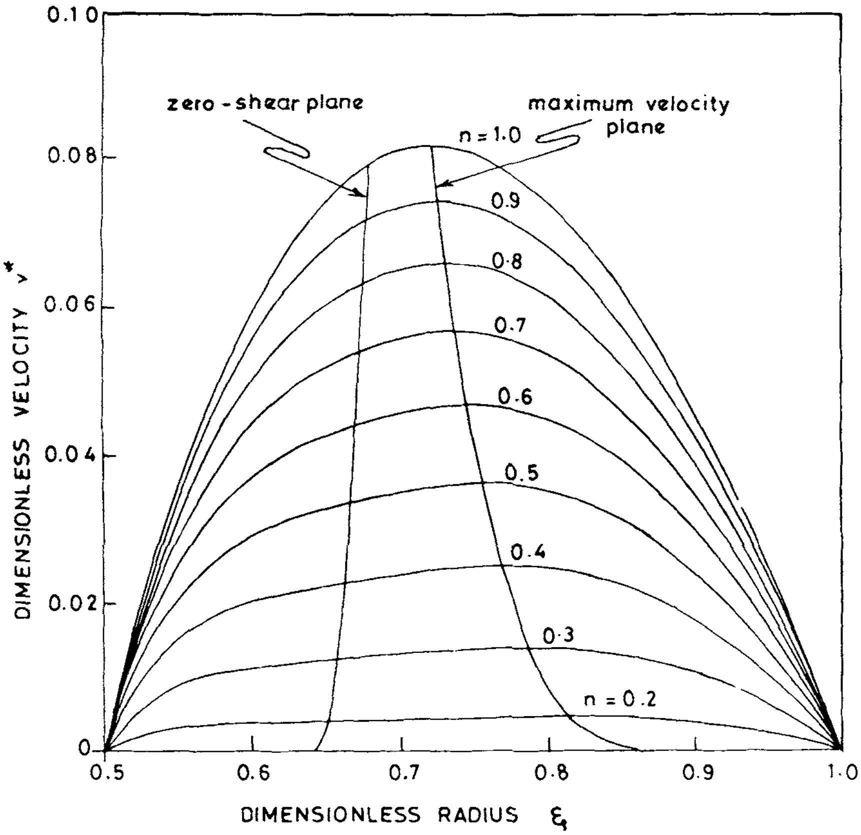

Figure 3 Velocity distributions for different n with $K = 0.5$ for tangential pressure flow.

NOTATION

K	Radius ratio of inner to outer cylinder
L	Length of cylinders
m	Consistency index in power-law model
n	Power-law index
P	Pressure
q	Parameter (0 for tangential flow and 1 for axial flow)
Q	Volumetric flow rate
R	Radius of outer cylinder
r	Radial distance in cylindrical coordinates
v_z	Local velocity in z-direction
v_θ	Local velocity in θ-direction
x	Dummy variable for integration
z	Axial distance in cylindrical coordinates
γ	Shear rate
γ_c	Characteristic shear rate as defined in Table I
θ	Tangential coordinate in cylindrical coordinate system
λ	Dimensionless zero-shear radius
μ	Viscosity of Newtonian fluid
ξ	Dimensionless distance in radial direction
τ_{rz}, $\tau_{r\theta}$	Components of stress tensor
*	Superscript denoting dimensionless quantity

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